TOPOLOGY OF SPACES OF HYPERBOLIC POLYNOMIALS AND COMBINATORICS OF RESONANCES

BY

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ABSTRACT

In this paper we study the topology of the strata, indexed by number partitions λ, in the natural stratification of the space of monic hyperbolic polynomials of degree n. We prove stabilization theorems for removing an independent block or an independent relation in λ. We also prove contractibility of the one-point compactifications of the strata indexed by a large class of number partitions, including $\lambda = (k^m, 1^r)$, for $m \geq 2$. Furthermore, we study the maps between the homology groups of the strata, induced by imposing additional relations (resonances) on the number partition λ, or by merging some of the blocks of λ.

1. Introduction

The space of monic hyperbolic polynomials of degree n is naturally stratified by fixing the multiplicities of the roots. In this paper we study the topology of these strata. The topological aspects of the spaces of polynomials with multiple roots have been extensively studied, see e.q., [1, 2, 10, 11, 13, 14]. As the general motivation we would like to mention the widely-branching program of studying the topological properties of certain subsets of some fixed function space, namely of the spaces of functions with singularities of some fixed type (also known as *discriminants*).

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Here we study the discriminants in the function spaces of all hyperbolic polynomial maps from $\mathbb{R}^n$ to $\mathbb{R}^n$, but there is a very large number of other important examples:

- spaces of knots, i.e., nonsingular imbeddings $S^1 \longrightarrow S^3$, e.g., see [12];
- spaces of complex polynomial maps, or, more generally, of systems of polynomials, see for example [13, Theorem 4, p. 126], and [4], for a connection between these two cases;
- spaces of nonsingular deformations of differentiable manifolds;

and many others. V. Vassiliev's book [13] provides a very extensive overview of the developments and the current state of the art in this area.

The case of hyperbolic polynomials has been considered before, most importantly in [10], where a simplicial complex of a combinatorial nature δ_λ was described, such that the double suspension of δ_λ is homeomorphic to $\widehat{\mathrm{Hyp}_\lambda^n}$. Here $\widehat{\mathrm{Hyp}_\lambda^n}$ is a one-point compactification of the strata of the space of all monic hyperbolic polynomials of degree n, which is indexed by the number partition λ; the exact definition is given in Subsection 2.1.

In this paper we are working further with this combinatorial model. By using the techniques of the Discrete Morse Theory as well as direct algebro-topological arguments, we are able to compute homology groups of $\widehat{\mathrm{Hyp}_\lambda^n}$ and even, in some cases, determine the homotopy type of $\widehat{\mathrm{Hyp}_\lambda^n}$, for several previously unknown classes of λ.

We would like to emphasize the combinatorial aspect at this point. The methods which we use are combinatorial. In fact, we are working exclusively with posets of compositions, which can be considered as objects of internal combinatorial interest, even though their appearance was mainly motivated by the topological questions about $\widehat{\mathrm{Hyp}_\lambda^n}$.

Boris Shapiro has suggested to me in a private conversation, [9], that there might exist a general algorithm for computing the homology groups (or even better, the homotopy type) of $\widehat{\mathrm{Hyp}_\lambda^n}$ for general λ. The results of this paper along with the idea of the resonance category, introduced in [15], may be used as the first step on this path.

Here is a short summary of the contents.

Section 2. Notations and terminology are introduced. The description of the Shapiro–Welker combinatorial model for $\widehat{\mathrm{Hyp}_\lambda^n}$ is given. The relevant results of Discrete Morse Theory are outlined for later use.

Section 3. Here the bulk of our results is concentrated. In Subsection 3.1, we prove the First Stabilization Theorem which allows one to remove an indepen-

dent block from a number partition λ. Furthermore, Theorem 3.2 describes, in particular, a large class of partitions λ (to which, for example, $\lambda = (k^m, 1^r)$, $m \geq 2$, belongs), for which $\widehat{\mathrm{Hyp}^n_\lambda}$ is contractible.

In Subsection 3.2, we prove the Second Stabilization Theorem which allows one in some situations to remove a relation from a partition. We also study the maps between simplicial complexes δ_λ which are induced by imposing additional relations (resonances) on the number partition λ, or by merging some of the blocks of λ.

Section 4.   We describe some remaining open questions and perspective developments.

2. Methods

2.1. COMPOSITIONS, NUMBER PARTITIONS, AND THE INDEXING OF STRATA. An ordered tuple of positive integers $\lambda = [\pi_1, \ldots, \pi_t]$ is called a **composition**, or sometimes a **composition of** n, where $n = \pi_1 + \cdots + \pi_t$. When this tuple is taken unordered, λ is called a **number partition** of n, we write $\lambda = (\pi_1, \ldots, \pi_t)$, and $\lambda \vdash n$. For number partitions, we also use the power notation:

$$(n^{\alpha_n}, \ldots, 1^{\alpha_1}) = (\underbrace{n, \ldots, n}_{\alpha_n}, \ldots, \underbrace{1, \ldots, 1}_{\alpha_1}).$$

Both for number partitions and compositions, we call π_i's the **blocks** of λ. The **length** of λ is the number of blocks and is denoted $l(\lambda)$. Given a composition of n, its **type** is the number partition of n, which is obtained from the composition by forgetting the order of the blocks (in the text we often reflect it by changing the square brackets to round ones).

Let $\mathrm{Hyp}^n \subseteq \mathbb{R}^n$ be the space of all monic hyperbolic polynomials (a polynomial is called hyperbolic when all of its roots, and hence its coefficients, are real numbers), and $\widehat{\mathrm{Hyp}^n}$ its one-point compactification. There is a standard cell decomposition of $\widehat{\mathrm{Hyp}^n}$ which we now proceed to describe. For a composition

$[\alpha_1, \ldots, \alpha_t]$ of n, we denote by $\mathrm{Hyp}^n_{[\alpha_1,\ldots,\alpha_t]}$ the topological space of all hyperbolic polynomials $(x - r_1)^{\alpha_1} \cdots (x - r_t)^{\alpha_t}$ such that $r_1 < \cdots < r_t$. Given a number partition λ of n, we denote by Hyp^n_λ the closure (in Hyp^n) of the union of all cells $\mathrm{Hyp}^n_{[\alpha_1,\ldots,\alpha_t]}$, where the composition $[\alpha_1, \ldots, \alpha_t]$ is of type λ. We denote the one-point compactification of Hyp^n_λ by $\widehat{\mathrm{Hyp}^n_\lambda}$.

2.2. THE SHAPIRO–WELKER MODEL. The set of all compositions of n is partially ordered by refinement. Namely, let $x = [\alpha_1, \ldots, \alpha_{l(x)}]$ and $y = [\beta_1, \ldots, \beta_{l(y)}]$ be two compositions of n; we say that $x \leq y$ if and only if $\alpha_j = \beta_{i_{j-1}+1} + \cdots + \beta_{i_j}$, for $1 \leq j \leq l(x)$, and some $0 = i_0 < i_1 < \cdots < i_{l(x)} = l(y)$. Since $\beta_i > 0$, for $i = 1, \ldots, l(y)$, the indices $i_1, \ldots, i_{l(x)-1}$ are uniquely defined. In this situation, we set $g(y, x, \beta_i) = j$ if and only if $i_{j-1} + 1 \leq i \leq i_j$.

Given a number partition $\lambda = (\pi_1, \ldots, \pi_t)$ of n, we define D_λ to be the poset consisting of all compositions of n which are less than or equal to some composition of n of type λ. D_λ has a minimal element, the composition consisting of just the number n, and it is easy to see that $D_\lambda \cup \{\hat{1}\}$ is a lattice, where $\hat{1}$ is an adjoint maximal element.

Since the lower intervals of D_λ are boolean algebras, and D_λ itself is meet-semilattice, there exists a unique simplicial complex, which we denote by δ_λ, such that D_λ is the face poset of δ_λ, i.e., the elements of D_λ and the simplices of δ_λ are in bijection, and the partial order relation on D_λ corresponds under this bijection to the inclusions of simplices of δ_λ. In particular, when $l(\lambda) = 1$, we have $\delta_\lambda = \emptyset$.

This bijection is the reason why we chose an order convention on the compositions as opposed to the customary one in combinatorics: we want to have an order-preserving bijection, not an order-reversing one.

The simplicial complex δ_λ is important for the following reason.

THEOREM 2.1 ([10, Theorem 3.5(a)]): *Let λ be a number partition of n. Then the one-point compactification of the strata indexed by λ, $\widehat{\mathrm{Hyp}^n_\lambda}$, is homeomorphic to the double suspension of the simplicial complex δ_λ.*

2.3. TERMINOLOGY OF RESONANCES. It turns out that δ_λ depends only on the set of various equalities of sums of different parts of λ (resonances), not on the exact numerical values of the parts of λ. This can be formalized as follows.

Definition 2.2: (a) Given a composition of n, $\alpha = [\pi_1, \ldots, \pi_{l(\alpha)}]$, a **resonance** of α is an unordered pair $(\{i_1, \ldots, i_k\}, \{j_1, \ldots, j_m\})$ of nonempty disjoint subsets

of $\{1, \ldots, l(\alpha)\}$, such that

(2.1) $$\pi_{i_1} + \cdots + \pi_{i_k} = \pi_{j_1} + \cdots + \pi_{j_m}.$$

We denote the set of all resonances of α by $\operatorname{Res} \alpha$.

(b) Given a number partition $\lambda \vdash n$, we denote by $\operatorname{Res} \lambda$ the set of all $\operatorname{Res} \alpha$, such that α has type λ.

For every positive integer k, the permutation action of $\mathcal{S}_k$ on $[k]$ induces an $\mathcal{S}_k$-action on the set of all compositions of length k, and hence also on the set $\{\operatorname{Res} \alpha \mid \alpha$ is a composition of length $k\}$. The orbits are indexed by number partitions, and $\operatorname{Res} \lambda$ is the orbit of this action indexed by λ. For example, $\operatorname{Res}(8, 3, 3, 3, 1) = \operatorname{Res}(7, 5, 4, 4, 4)$.

The notion of a resonance is important, because if λ and $\tilde{\lambda}$ have the same set of resonances, then the spaces $\widehat{\operatorname{Hyp}_\lambda^n}$ and $\widehat{\operatorname{Hyp}_{\tilde{\lambda}}^n}$ are homeomorphic.

We will abuse the language and call an equality of the form (2.1) itself a resonance. We will also say that α (or λ) has this resonance. Where it does not lead to confusion, we shall often say "the set of resonances of the number partition λ", instead of saying "the set of resonances of some composition of type λ". If in (2.1), $k = m = 1$, then the resonance is called **trivial**, otherwise it is called **nontrivial**. The resonance $\pi_{i_1} + \cdots + \pi_{i_k} = \pi_{j_1} + \cdots + \pi_{j_m}$ is said **to involve** blocks $\pi_{i_1}, \ldots, \pi_{i_k}, \pi_{j_1}, \ldots, \pi_{j_m}$; correspondingly, these blocks are said **to be involved** in this resonance.

A block π_i of a composition is called **independent** if, whenever $\pi_{i_1} + \cdots + \pi_{i_k} = \pi_{j_1} + \cdots + \pi_{j_m}$ and $i_1 = i$, there exists $1 \leq q \leq m$ such that there is a trivial resonance $\pi_i = \pi_{j_q}$. If a block is not involved in any resonance then it is called **strongly independent**.

Clearly, given a block in a number partition λ, the corresponding block is independent, resp. strongly independent, in all compositions of type λ if and only if it is independent, resp. strongly independent, in any one such composition. Therefore, we have a well-defined notion for a block of a number partition to be independent, resp. strongly independent.

If $\lambda = (\pi, \pi_1, \ldots, \pi_{l(\lambda)-1})$ is a number partition such that π is strongly independent, then for any $x \in D_\lambda$ there exists a number $1 \leq \rho_\lambda(x, \pi) \leq l(x)$ such that, whenever $y \geq x$ and y is of type λ, we have $\rho_\lambda(x, \pi) = g(y, x, \pi)$, i.e., this number does not depend on the choice of y.

More generally, if π is independent, but not necessarily strongly independent: say the trivial resonances in which it is involved are $\pi = \pi_1, \ldots, \pi = \pi_k$, then the multiset $\{g(y, x, \pi_i) \mid i = 1, \ldots, k\} \cup \{g(y, x, \pi)\}$ does not depend on the

choice of the composition y of type λ, such that $y \geq x$. In such a situation, if $x = [\alpha_1, \ldots, \alpha_{l(x)}]$, we say that α_j **contains the block** π, if j belongs to the multiset $\{g(y, x, \pi_i) | \ i = 1, \ldots, k\} \cup \{g(y, x, \pi)\}$. We refer to the number of occurrences of j in this multiset as the number of copies of π contained in α_j.

2.4. DISCRETE MORSE THEORY.   For a regular CW complex δ, we denote by $P(\delta)$ its face poset, the empty face included as a minimal element. Vice versa, for a poset P, $\delta(P)$ denotes the simplicial complex which is the nerve (order complex) of P; see [7, 8] for the first appearances of the nerve of a category. Recall also the following terminology.

Definition 2.3:   Let X be a regular CW complex. Assume that F_1 and F_2 are cells of X such that F_2 is a maximal cell which contains F_1, and there is no other maximal cell containing F_1. A **collapse** is the replacement of X with $X \smallsetminus \{F| \ F_1 \subseteq F\}$. A collapse is called **elementary** if $\dim F_1 + 1 = \dim F_2$.

Clearly, a collapse is a strong deformation retract, hence it preserves the homotopy type of the space.

Let δ be a regular CW complex. A **matching** W on $P = P(\delta)$ (cf. [5, Definition 9.1]) is a set of disjoint pairs (σ, τ) such that $\tau, \sigma \in P$, $\tau \succ \sigma$, ("$\succ$" denotes the covering relation). We set

$$\overrightarrow{W} = \{\sigma \in P \,|\, \text{there exists } \tau \text{ such that } (\sigma, \tau) \in W\},$$
$$\overleftarrow{W} = \{\tau \in P \,|\, \text{there exists } \sigma \text{ such that } (\sigma, \tau) \in W\}.$$

If $(\sigma, \tau) \in W$ then we set $W(\sigma) = \tau$.

Definition 2.4 (cf. [5, Definition 9.2]):   A matching is called **acyclic** if it is impossible to find a sequence $\sigma_0, \ldots, \sigma_t \in \overrightarrow{W}$ such that $\sigma_0 \neq \sigma_1$, $\sigma_0 = \sigma_t$ and $W(\sigma_i) \succ \sigma_{i+1}$ for $0 \leq i \leq t - 1$.

Note that if a matching is acyclic, then not all 0-cells are matched with 1-cells.

A cell σ is called **critical** if either it is the empty cell, or it is a 0-cell matched with the empty cell, or $\sigma \notin \overleftarrow{W} \cup \overrightarrow{W}$; in the latter case σ is called **nontrivial critical**. Let $m_i(W)$ denote the number of critical i-cells.

Note: Alternatively, we could just omit the empty cell from the cell complex. However, we choose to keep it, and have a somewhat more complicated definition of the critical cells, first, since the empty cell is natural in our applications, and second, since the formulation of the Theorem 2.5 is somewhat smoother in that version, as a complete matching is a standard object in combinatorics.

We need the following result; see also [6, Theorem 3.2], [5, Theorem 9.3], and [5, Corollary 3.5].

THEOREM 2.5: *Let δ be a regular CW complex of dimension d, and let W be an acyclic matching on $P(\delta)$. Then δ is homotopy equivalent to a CW complex δ^M, which has $m_i(W)$ cells of dimension i. In particular, if the acyclic matching is complete, then δ is contractible.*

The basic idea of the proof is that the combinatorial condition of acyclicity allows us to arrange the collapses in a sequence and perform them one after the other.

3. Results

3.1. APPLICATIONS OF DISCRETE MORSE THEORY.   We begin by proving a theorem which allows one to remove an independent block from a number partition.

FIRST STABILIZATION THEOREM 3.1: *Let $\lambda = (\pi_1, \ldots, \pi_t)$ be a number partition of n, such that π_1 is independent.*

  (a) *If π_1 is not strongly independent, i.e., $\pi_1 = \pi_i$, for some $2 \leq i \leq t$, then the simplicial complex δ_λ, and therefore also the topological space $\widehat{\mathrm{Hyp}^n_\lambda}$, is contractible.*

  (b) *If, on the other hand, $\pi_1 \neq \pi_i$, for all $2 \leq i \leq t$, then δ_λ is homotopy equivalent to $\mathrm{susp}\, \delta_{\tilde\pi}$, correspondingly $\widehat{\mathrm{Hyp}^n_\lambda} \simeq \mathrm{susp}\, \widehat{\mathrm{Hyp}^n_{\tilde\pi}}$, where $\tilde\pi = (\pi_2, \ldots, \pi_t)$.*

Proof: (a) For any $a = [\alpha_1, \ldots, \alpha_{l(a)}] \in D_\lambda$ let $\iota(a)$, resp. $\gamma(a)$, be the smallest index of a block of a which is equal to π_1, resp. larger than π_1 and containing at least one copy of π_1, if such exists; otherwise put $\iota(a)$, resp. $\gamma(a)$, equal to ∞. Clearly at least one of the numbers $\iota(a)$ and $\gamma(a)$ is finite. Since $\iota(a) \neq \gamma(a)$, the elements of D_λ split into two disjoint sets: $A = \{a \in D_\lambda | \, \iota(a) > \gamma(a)\}$ and $B = \{a \in D_\lambda | \, \iota(a) < \gamma(a)\}$.

Consider the following matching W on D_λ. For

$$a = [\alpha_1, \ldots, \alpha_{l(a)}], \quad b = [\beta_1, \ldots, \beta_{l(b)}] \in D_\lambda,$$

$a \in A$, $b \in B$, we have $(a, b) \in W$ if and only if $l(a) + 1 = l(b)$, and

$$\begin{cases} \alpha_i = \beta_i, & \text{for } i = 1, \ldots, \iota(b) - 1; \\ \alpha_{\iota(b)} = \beta_{\iota(b)} + \beta_{\iota(b)+1}; \\ \alpha_i = \beta_{i+1}, & \text{for } i = \iota(b) + 1, \ldots, l(a). \end{cases}$$

Note that in such a case $\iota(b) = \gamma(a)$ and $\iota(b) \le l(b) - 1$.

Given $a \in A$, by the definition of W, there exists a uniquely defined $b \in B$ such that $(a, b) \in W$; conversely, given $b \in B$, one obtains a unique a such that $(a, b) \in W$ by breaking the block $\beta_{\gamma(b)}$ into the blocks $\pi_1, \beta_{\gamma(b)} - \pi_1$ (exactly in this order).

It is clear that these procedures are inverses of each other, thus W is a well-defined matching. Furthermore, W is complete. Note that the empty simplex $[\pi_1 + \cdots + \pi_t]$ is matched with the vertex $[\pi_1, \pi_2 + \cdots + \pi_t]$. We have $\overrightarrow{W} = A$ and $\overleftarrow{W} = B$.

The topologically inclined reader may think of this matching as a set of elementary collapses. We shall now check that they in fact can be arranged in a sequence of collapses, by checking that W is an acyclic matching. Consider a sequence $a_0, \ldots, a_k \in A$, such that $a_0 = a_k$, $a_i \ne a_{i+1}$ and $W(a_i) \succ a_{i+1}$, for $i = 0, \ldots, k-1$. The simple but crucial observation is that $\iota(W(a_i)) > \gamma(a_{i+1})$.

Indeed, since $a_{i+1} \in A$ and $W(a_i) \in B$, we have $\gamma(a_{i+1}) < \iota(a_{i+1})$ and $\gamma(W(a_i)) > \iota(W(a_i))$. Therefore, there are two cases. The first one is when a_{i+1} is obtained from $W(a_i)$ by merging blocks with indices $\iota(W(a_i)) - 1$ and $\iota(W(a_i))$, in which case $\gamma(a_{i+1}) = \iota(W(a_i)) - 1$. The second case is when a_{i+1} is obtained from $W(a_i)$ by merging blocks with indices $j-1$ and j, such that $j < \iota(W(a_i))$, and the obtained block is larger than π_1 and contains at least one copy of π_1, in which case $\gamma(a_i) = j - 1 < \iota(W(a_i))$. Note that we are using the fact that $a_i \ne a_{i+1}$ by ruling out the possibility that a_{i+1} is obtained from $W(a_i)$ by merging the blocks indexed $\iota(W(a_i))$ and $\iota(W(a_i)) + 1$.

Since, as observed before, $\gamma(a_i) = \iota(W(a_i))$, we obtain $\gamma(a_i) > \gamma(a_{i+1})$ and hence a contradiction $\gamma(a_0) > \gamma(a_1) > \ldots > \gamma(a_t) = \gamma(a_0)$. Thus, by Theorem 2.5, the simplicial complex δ_λ is contractible.

(b) Let x be the vertex of δ_λ which is labeled by the composition $[\pi_2 + \cdots + \pi_t, \pi_1]$. Consider the following two subspaces of δ_λ, $T = \{\text{closed star}_{\delta_\lambda}(x)\}$ and $Q = \delta_\lambda \setminus \{\text{open star}_{\delta_\lambda}(x)\}$. Clearly T is contractible and $\delta_\lambda = T \cup Q$.

Let us show that Q is contractible. Consider the matching W on δ_λ which is defined completely analogously to the one in the first part of this proof. It is easy to see that the matching is again complete. The acyclicity of W follows from the argument in the first part.

Then δ_λ is homotopy equivalent to the suspension of $T \cap Q = \text{link}_{\delta_\lambda}(x)$. The simple observation that $\text{link}_{\delta_\lambda}(x) = \delta_{(\pi_2, \ldots, \pi_t)}$ finishes the proof. ∎

Shapiro and Welker have computed the homotopy type of δ_λ for $\lambda = (k, 1^r)$, [10, Proposition 3.9, Corollary 3.10], by using the previous work of Björner and

Wachs, [3, Theorem 8.2, Corollary 8.4]. Here we prove a general theorem of which $\lambda = (k^m, 1^r)$, for $m \geq 2$, is a special case. We shall also reprove the result for the case $\lambda = (k, 1^r)$.

First we need some terminology. Number partitions of n are partially ordered by refinement: for $\lambda, \mu \vdash n$, $\lambda = (\pi_1, \ldots, \pi_{l(\lambda)})$, $\mu = (\eta_1, \ldots, \eta_{l(\mu)})$, we say that $\lambda \geq \mu$ if there exists a collection of disjoint sets $I_1, \ldots, I_{l(\mu)} \subseteq \{1, \ldots, l(\lambda)\}$, such that $\bigcup_{i=1}^{l(\mu)} I_i = \{1, \ldots, l(\lambda)\}$ and $\sum_{j \in I_k} \pi_j = \eta_k$, for $k = 1, \ldots, l(\mu)$. Again, the convention of the ordering is dictated by the partial order on the set of compositions, which in turn followed the pattern of cell inclusions.

Cutting Condition.

We say that a pair (λ, π_1), where $\lambda = (\pi_1, \ldots, \pi_{l(\lambda)})$ is a number partition of n, satisfies the Cutting Condition, if, whenever $\mu = (\eta_1, \ldots, \eta_{l(\mu)})$ is another number partition of n, such that $\lambda \geq \mu$, and for some $i \in \{1, \ldots, l(\mu)\}$ and some nonempty set $I \subseteq \{2, \ldots, l(\lambda)\}$, we have equality $\eta_i = \pi_1 + \sum_{j \in I} \pi_j$, then we have $\lambda \geq \tilde{\mu}$, where $\tilde{\mu} = (\eta_1, \ldots, \eta_{i-1}, \pi_1, \sum_{j \in I} \pi_j, \eta_{i+1}, \ldots, \eta_{l(\mu)})$.

Note that if (λ, π_1) satisfies the cutting condition, then the block π_1 is not necessarily the largest one. For example, $((6, 4, 4, 2, 1), 1)$ satisfies the cutting condition while $((6, 4, 4, 2, 1), 6)$ does not: $(6, 4, 4, 2, 1) > (8, 6, 2, 1)$, but $(6, 4, 4, 2, 1) \not\geq (6, 6, 2, 2, 1)$.

THEOREM 3.2: *Let $\lambda = (\pi_1, \ldots, \pi_{l(\lambda)})$ be a number partition, such that (λ, π_1) satisfies the cutting condition.*

(a) *If π_1 is involved in a trivial resonance, then δ_λ is contractible, and hence so is $\widehat{\mathrm{Hyp}_\lambda^n}$.*

(b) *If $\pi_1 \neq \pi_i$, for $i = 2, \ldots, l(\lambda)$, then there exists an acyclic matching on D_λ such that the nontrivial critical simplices are exactly the ones indexed by the compositions $a = (\alpha_1, \ldots, \alpha_{l(a)})$, where $\alpha_{l(a)} = \pi_1$, and $\alpha_i \neq \pi_1 + \sum_{j \in I} \pi_j$ for all $1 \leq i \leq l(a) - 1$ and $I \subseteq \{2, \ldots, l(\lambda)\}$ (I may be empty).*

Proof: We can define a matching on D_λ which is completely analogous to the one defined in the proof of the Stabilization Theorem 3.1(a). A word by word check of the proof of the acyclicity of the matching reveals that the same argument is still valid in our case.

If $\pi_1 \neq \pi_i$, for $i = 2, \ldots, l(\lambda)$, the nontrivial critical simplices are the simplices corresponding to the compositions described in (b) above, because they are the only ones where, on the one hand, π_1 cannot be merged with the next block to the right, and, on the other hand, it is impossible to cut off a block π_1 from some other block. ∎

Theorem 3.1(a) follows from Theorem 3.2(a). However, since the proofs are essentially identical, we prefer to prove the structural Stabilization Theorem 3.1 first, and then point out that the argument is actually valid for the combinatorially more technical Theorem 3.2. The next result shows that Theorem 3.2 is strictly more general than Theorem 3.1.

COROLLARY 3.3: *Let*

$$\lambda = \big(\underbrace{k^{c_1}, \ldots, k^{c_1}}_{m_1}, \ldots, \underbrace{k^{c_t}, \ldots, k^{c_t}}_{m_t}, \underbrace{1, \ldots, 1}_{m_{t+1}}\big),$$

for some positive integers $k, m_1, \ldots, m_{t+1}$, and $c_1 > \cdots > c_t$, such that $k \geq 2$, and $m_1 \geq 2$. Then δ_λ, and hence also $\widehat{\mathrm{Hyp}^n_\lambda}$, is contractible.

Proof: It is enough to check that (λ, k^{c_1}) satisfies the cutting condition, since then, by Theorem 3.2(a), the simplicial complex δ_λ is contractible.

Assume $\mu = (\eta_1, \ldots, \eta_{l(\mu)})$, $\lambda > \mu$, and $\eta_q = \sum_{i=1}^t r_i k^{c_i} + r_{t+1}$ for $r_i \leq m_i$, $r_1 \geq 1$, and either $r_1 \geq 2$ or $r_i > 0$ for some $i \in \{2, \ldots, t+1\}$. Since $\lambda \geq \mu$, one can write $\eta_j = \sum_{i=1}^t r_{i,j} k^{c_i} + r_{t+1,j}$, for $j = 1, \ldots, l(\mu)$, so that $\sum_{j=1}^{l(\mu)} r_{i,j} = m_i$, for $i = 1, \ldots, t+1$. If $r_{1,q} \geq 1$, then we are done. Otherwise, as $\eta_q > k^{c_1}$, we can find $\tilde{r}_i \leq r_{i,q}$, for $i = 2, \ldots, t+1$, such that $\sum_{i=2}^t \tilde{r}_i k^{c_i} + \tilde{r}_{t+1} = k^{c_1}$. This means that there exist numbers $\tilde{r}_{i,j}$ such that $\eta_j = \sum_{i=1}^t \tilde{r}_{i,j} k^{c_i} + \tilde{r}_{t+1,j}$, for $j = 1, \ldots, l(\mu)$, $\sum_{j=1}^{l(\mu)} \tilde{r}_{i,j} = m_i$, for $i = 1, \ldots, t+1$, and $\tilde{r}_{1,q} \geq 1$.   ∎

There are many other pairs satisfying the cutting condition, for example:
- $((\pi^p, \pi_1^{q_1}, \ldots, \pi_m^{q_m}, 1^r), \pi)$, for $\pi \geq \sum_{i=1}^m q_i \pi_i$;
- $((k\pi_1, \ldots, k\pi_t, 1^r), 1)$, for $1 \leq r \leq k-1$.

Often the matching produced in Theorem 3.2(b) can be extended so as to yield complete information on the homology groups of the simplicial complex δ_λ. In the next proposition we demonstrate this by a couple of examples.

PROPOSITION 3.4:
(a) [10, Proposition 3.9, Corollary 3.10]: *For $\lambda = (k, 1^t)$, where $k \geq 2$, $t \geq 0$, we have*

$$\delta_\lambda \simeq \begin{cases} S^{2m-1}, & \text{if } t = km, \text{ for some } m \in \mathbb{Z}; \\ S^{2m}, & \text{if } t = km+1, \text{ for some } m \in \mathbb{Z}; \\ point, & otherwise. \end{cases}$$

(b) *For $\lambda = (k, 2, 1^t)$, where $k \geq 3$, we have*

$$\widetilde{H}_*(\delta_\lambda) = \begin{cases} \mathbb{Z}_{(t)} \oplus \mathbb{Z}_{(2m-1)}, & \text{if } t+2 = km, \text{ for some } m \in \mathbb{Z}; \\ \mathbb{Z}_{(t)} \oplus \mathbb{Z}_{(2m)}, & \text{if } t+1 = km, \text{ for some } m \in \mathbb{Z}; \\ \mathbb{Z}_{(t)}, & otherwise. \end{cases}$$

Recall that $\widehat{\mathrm{Hyp}^n_\lambda} \cong \mathrm{susp}^2 \delta_\lambda$, hence (a) and (b) above yield the corresponding information about the stratum $\widehat{\mathrm{Hyp}^n_\lambda}$.

Note: In this paper, all homology groups are reduced and with integer coefficients. We also use the notation $\mathbb{Z}_{(i)}$ to denote a direct summand $\mathbb{Z}$ in the ith reduced homology group. For example, the reduced homology groups of the torus $S^1 \times S^1$ would be written as $\mathbb{Z}_{(1)} \oplus \mathbb{Z}_{(1)} \oplus \mathbb{Z}_{(2)}$.

Proof of Proposition 3.4:   (a) Let us extend the matching given in the proof of Theorem 3.2(b) (equivalently, in the proof of Stabilization Theorem 3.1) as follows. If $a \in D_\lambda$ indexes a nontrivial critical simplex and

$$a \neq [1, k-1, 1, \ldots, k-1, 1, k], \quad a \neq [1, k-1, \ldots, 1, k-1, k],$$

then

$$a = [\underbrace{1, k-1, \ldots, 1, k-1}_{2m}, p, q, \ldots, k],$$

where $m \geq 0$, $p \leq k-1$, and, either $p \geq 2$ or $q \leq k-2$.

If $p \geq 2$, we define

$$W(a) = [\underbrace{1, k-1, \ldots, 1, k-1}_{2m}, 1, p-1, q, \ldots, k].$$

If $p = 1$ and $q \leq k-2$, we have $a = W(b)$ for

$$b = [\underbrace{1, k-1, \ldots, 1, k-1}_{2m}, 1+q, \ldots, k].$$

This will complement the existing matching so that the only remaining nontrivial critical simplices are

$$[\underbrace{1, k-1, 1, \ldots, k-1}_{2m}, 1, k], \text{ if } t = km+1, \text{ and } [\underbrace{1, k-1, \ldots, 1, k-1}_{2m}, k], \text{ if } t = km.$$

It only remains to check that W is still acyclic. Since the newly matched simplices form an upper ideal of D_λ, it is enough to check the acyclicity condition involving only them. Let $a_0, \ldots, a_f \in D_\lambda$ be such that $a_0 = a_f$, $a_i \neq a_{i+1}$,

$$a_i = [\underbrace{1, k-1, \ldots, 1, k-1}_{2m_i}, p_i, q_i, \ldots, k],$$

$k-1 \geq p_i \geq 2$, and

$$W(a_i) = [\underbrace{1, k-1, \ldots, 1, k-1}_{2m_i}, 1, p_i-1, q_i, \ldots, k] \succ a_{i+1},$$

for $i = 0, \ldots, f - 1$.

Then, by what we have just said, a_{i+1} is obtained from $W(a_i)$ by merging the blocks indexed j and $j + 1$, for $j \geq 2m_i + 2$. If $j \geq 2m_i + 3$, or $j = 2m_i + 2$, but $p_i - 1 + q_i \neq k - 1$, we get into a contradiction with the choice of a_i's. Hence we must have $j = 2m_i + 2$ and $p_i - 1 + q_i = k - 1$, which implies that $m_i < m_{i+1}$, and we get a contradiction $m_0 < m_1 < \cdots < m_f = m_0$.

(b) The case $\lambda = (k, 2, 1^t)$ is very similar. The only difference is that there is an additional nontrivial critical t-simplex

$$[2, \underbrace{1, \ldots, 1}_{t}, k]$$

(according to the idea of the previous matching, one would want to break 2 into $1, 1$, which is impossible). Thus, from the previous argument we derive the conclusion, unless the nontrivial critical cells (for the case $t + 2 = km$ and $t + 1 = km$) are in the neighboring dimensions. These cases are $(4, 2, 1, 1)$, $(4, 2, 1, 1, 1)$, $(3, 2, 1^4)$, and $(3, 2, 1^5)$; they can be verified directly. Observe that since for some cases we obtain a homotopy equivalence of δ_λ with a CW complex with 2 cells in dimensions higher than 0, we cannot in general determine the homotopy type of δ_λ. ∎

Note that since we are not using [3, Theorem 8.2, Corollary 8.4] for our proof of Proposition 3.4(a), we obtain the alternative proof of these results of Björner and Wachs on the homotopy type of the lattice of intervals generated by all $(k - 1)$-element subsets of $\{1, \ldots, n - 1\}$.

3.2. IDENTIFICATIONS CAUSED BY ADDITIONAL RESONANCES. Let us introduce one more piece of terminology. Recall that, given a composition, its resonances are simply linear dependencies of its parts with coefficients $\pm 1, 0$, which we viewed as a pair of the subsets of the index set: those parts which get a coefficient 1 and those which get a coefficient -1. We say that a resonance r can be **derived** from a set of other resonances S if it follows from them as a linear equation. That is, if the entries in an integer vector satisfy the equations from the set S, then they also satisfy the equation r. A standard linear-algebraic way to see it is to pass to the dual vector space and view resonances there as vectors with $\pm 1, 0$ coordinates. Then r can be derived from S if and only if it lies on the linear span of vectors from S. If the resonance r cannot be derived from the set of resonances S, we say that r is **independent** of S.

Having this picture in mind, one can talk about adding independent resonances to the already existing set. If one could in general describe what happens to the

topology of the corresponding stratum, then one would have a general algorithm to compute the algebraic invariant of $\widehat{\mathrm{Hyp}^n_\lambda}$. Unfortunately, the combinatorics of the situation seems prohibitively complex.

In this subsection we take a small step on this road. Let us start with a simple observation.

PROPOSITION 3.5: *Let $\alpha = [\alpha_1, \ldots, \alpha_t]$ and $\tilde{\alpha} = [\tilde{\alpha}_1, \ldots, \tilde{\alpha}_t]$, $t \geq 3$, be two compositions such that the set of resonances of $\tilde{\alpha}$ is equal to the union of the set of resonances of α with a new resonance $\tilde{\alpha}_1 + \cdots + \tilde{\alpha}_k = \tilde{\alpha}_{k+1} + \cdots + \tilde{\alpha}_t$, for some $1 \leq k \leq t - 1$. Let $\lambda = (\alpha_1, \ldots, \alpha_t)$ and $\tilde{\lambda} = (\tilde{\alpha}_1, \ldots, \tilde{\alpha}_t)$, then $\widetilde{H}_*(\delta_{\tilde{\lambda}}) = \widetilde{H}_*(\delta_\lambda) \oplus \mathbb{Z}_{(1)}$, correspondingly $\widetilde{H}_*(\widehat{\mathrm{Hyp}^n_{\tilde{\lambda}}}) = \widetilde{H}_*(\widehat{\mathrm{Hyp}^n_\lambda}) \oplus \mathbb{Z}_{(3)}$.*

Note: The condition on the sets of resonances of the compositions α and $\tilde{\alpha}$ in the formulation of Theorem 3.5 is much stronger that just requiring that the resonance $\sum_{i=1}^k \tilde{\alpha}_i = \sum_{j=k+1}^t \tilde{\alpha}_j$ is independent of the resonances of α. It means that this is the *only* resonance added. For example, it implies the following: α has no resonance of the type $\sum_{i \in I} \alpha_i = \sum_{j \in J} \alpha_j$, for nonempty sets I and J, such that simultaneously $I \subseteq \{1, \ldots, k\}$ and $J \subseteq \{k+1, \ldots, t\}$.

Proof: Clearly, $\delta_{\tilde{\lambda}}$ is obtained from δ_λ by gluing together two vertices indexed by the compositions $[\alpha_1 + \cdots + \alpha_k, \alpha_{k+1} + \cdots + \alpha_t]$ and $[\alpha_{k+1} + \cdots + \alpha_t, \alpha_1 + \cdots + \alpha_k]$. Since the topological space δ_λ is connected, when $l(\lambda) \geq 3$, the result follows from the homology long exact sequence of a pair.   ∎

Next we consider the case when the added resonance does not include all the blocks, but is still the only resonance added.

SECOND STABILIZATION THEOREM 3.6: *Let $[\alpha_1, \ldots, \alpha_p, \beta_1, \ldots, \beta_q, \gamma_1, \ldots, \gamma_r]$ be a composition of type λ, such that $p \geq 1$, $q \geq 2$, $r \geq 1$, whose set of resonances includes*

$$(3.1) \qquad\qquad \beta_1 + \cdots + \beta_q = \gamma_1 + \cdots + \gamma_r.$$

Assume there exists $[\tilde{\alpha}_1, \ldots, \tilde{\alpha}_p, \tilde{\beta}_1, \ldots, \tilde{\beta}_q, \tilde{\gamma}_1, \ldots, \tilde{\gamma}_r]$, a composition of type $\tilde{\lambda}$, such that it has exactly the same resonances as λ, except for (3.1) (in particular, (3.1) is independent of the other resonances of λ). Assume also that the block $\gamma = \beta_1 + \cdots + \beta_q + \gamma_1 + \cdots + \gamma_r$ is strongly independent in $(\alpha_1, \ldots, \alpha_p, \gamma)$.
Then there exists a long exact sequence

$$\cdots \longrightarrow \widetilde{H}_{i-2}(\delta_{(\alpha_1, \ldots, \alpha_p)}) \xrightarrow{d_i} \widetilde{H}_i(\delta_{\tilde{\lambda}}) \longrightarrow \widetilde{H}_i(\delta_\lambda) \longrightarrow \widetilde{H}_{i-3}(\delta_{(\alpha_1, \ldots, \alpha_p)}) \xrightarrow{d_{i-1}} \cdots.$$

If, furthermore, $\tilde{\beta}_1$ and $\tilde{\beta}_2$ are independent in $\tilde{\lambda}$, then $d_i = 0$, and hence $\widetilde{H}_i(\delta_\lambda)$ can be found by solving the corresponding extension problem. In particular, if $\widetilde{H}_{i-3}(\delta_{(\alpha_1,\ldots,\alpha_p)})$ is free then $\widetilde{H}_i(\delta_\lambda) = \widetilde{H}_i(\delta_{\tilde{\lambda}}) \oplus \widetilde{H}_{i-3}(\delta_{(\alpha_1,\ldots,\alpha_p)})$.

Note 1: The corresponding information about $\widehat{\mathrm{Hyp}^n_\lambda}$ can be derived via the formula $\widehat{\mathrm{Hyp}^n_\lambda} \cong \mathrm{susp}^2 \delta_\lambda$.

Note 2: The case $q \geq 1$, $r \geq 2$ is symmetric to the case considered in Theorem 3.6, hence the same conclusion can be reached with β's and γ's interchanged. If $q = r = 1$, then the simplicial complex δ_λ is contractible by Stabilization Theorem 3.1.

Proof: Let A (resp. $\widetilde{A}$) be the simplicial subcomplex of $\delta_{\tilde{\lambda}}$ consisting of the simplices which are labeled by those compositions, where the sets of blocks $\{\tilde{\beta}_1, \ldots, \tilde{\beta}_q\}$ and $\{\tilde{\gamma}_1, \ldots, \tilde{\gamma}_r\}$ of $\tilde{\lambda}$ are summed up and the sum of $\tilde{\beta}_1, \ldots, \tilde{\beta}_q$ is either in the same block as the sum of $\tilde{\gamma}_1, \ldots, \tilde{\gamma}_r$ or to the left (resp. right) of it. Clearly, $B = A \cap \widetilde{A}$ is the simplicial subcomplex of $\delta_{\tilde{\lambda}}$, where all the blocks $\tilde{\beta}_1, \ldots, \tilde{\beta}_q, \tilde{\gamma}_1, \ldots, \tilde{\gamma}_r$ are summed up, and $A \cup \widetilde{A}$ is the simplicial subcomplex of $\delta_{\tilde{\lambda}}$, where the sets of blocks $\{\tilde{\beta}_1, \ldots, \tilde{\beta}_q\}$ and $\{\tilde{\gamma}_1, \ldots, \tilde{\gamma}_r\}$ of $\tilde{\lambda}$ are summed up.

There is a simplicial map $\delta_{\tilde{\lambda}} \longrightarrow \delta_\lambda$ which corresponds to imposing a new resonance $\sum_{i=1}^q \tilde{\beta}_i = \sum_{j=1}^r \tilde{\gamma}_j$ on $\tilde{\lambda}$. Topologically, it corresponds to gluing the subcomplexes A and $\widetilde{A}$ together in the natural way. There is a simplicial bijection ϕ between A and $\widetilde{A}$, which interchanges the sums $\sum_{i=1}^q \tilde{\beta}_i$ and $\sum_{j=1}^r \tilde{\gamma}_j$. This bijection fixes B and therefore we can glue A together with $\widetilde{A}$ inside $\delta_{\tilde{\lambda}}$ by pointwise identifying those simplices which are mapped to each other by ϕ. Gluing together two points in the proof of Proposition 3.5 is a simple special case of this procedure.

Let $\bar{A}$ denote the simplicial subcomplex of δ_λ consisting of simplices indexed by compositions where the sets $\{\beta_1, \ldots, \beta_q\}$ and $\{\gamma_1, \ldots, \gamma_r\}$ are summed up. We have $\bar{A} = \delta_{(\alpha_1,\ldots,\alpha_p,\beta,\beta)}$, where $\beta = \beta_1 + \cdots + \beta_q = \gamma_1 + \cdots + \gamma_r$. The block β must be independent in $(\alpha_1, \ldots, \alpha_p, \beta, \beta)$.

Indeed, since γ is strongly independent in $(\alpha_1, \ldots, \alpha_p, \gamma)$, there is no resonance of the type $\sum_{i \in I} \alpha_i + \beta + \beta = \sum_{j \in J} \alpha_j$. The only other option for β *not* to be independent would be to have a resonance of the type $\sum_{i \in I} \alpha_i + \beta = \sum_{j \in J} \alpha_j$. But then λ would have resonances $\sum_{i \in I} \alpha_i + \sum_{i=1}^q \beta_i = \sum_{j \in J} \alpha_j$ and $\sum_{i \in I} \alpha_i + \sum_{j=1}^r \gamma_j = \sum_{j \in J} \alpha_j$. These two resonances imply (3.1), which contradicts the existence of $\tilde{\lambda}$.

So β is independent, and hence, by Stabilization Theorem 3.1(a), $\bar{A}$ is contractible. Furthermore, it is clear that A, $\widetilde{A}$ and $\bar{A}$ are all isomorphic, hence they

are all contractible.

By the very nature of the gluing map $\delta_{\tilde{\lambda}} \longrightarrow \delta_{\lambda}$ we have a simplicial isomorphism of pairs $(\delta_{\tilde{\lambda}}, A \cup \widetilde{A}) \cong (\delta_{\lambda}, \bar{A})$; for the general criteria see Proposition 4.1. Combining this observation with a long exact sequence

$$\cdots \longrightarrow \widetilde{H}_i(\bar{A}) \longrightarrow \widetilde{H}_i(\delta_{\lambda}) \longrightarrow \widetilde{H}_i(\delta_{\lambda}, \bar{A}) \longrightarrow \widetilde{H}_{i-1}(\bar{A}) \longrightarrow \cdots,$$

we conclude that $\widetilde{H}_i(\delta_{\lambda}) = \widetilde{H}_i(\delta_{\lambda}, \bar{A}) = \widetilde{H}_i(\delta_{\tilde{\lambda}}, A \cup \widetilde{A})$. We also have a long exact sequence

$$\cdots \longrightarrow \widetilde{H}_i(A \cup \widetilde{A}) \xrightarrow{d_i} \widetilde{H}_i(\delta_{\tilde{\lambda}}) \longrightarrow \widetilde{H}_i(\delta_{\tilde{\lambda}}, A \cup \widetilde{A}) \longrightarrow \widetilde{H}_{i-1}(A \cup \widetilde{A}) \xrightarrow{d_{i-1}} \cdots.$$

Since both A and $\widetilde{A}$ are contractible, we have

$$\widetilde{H}_i(A \cup \widetilde{A}) = \widetilde{H}_{i-1}(A \cap \widetilde{A}) = \widetilde{H}_{i-1}(B).$$

Clearly $B = \delta_{(\alpha_1,\dots,\alpha_p,\gamma)}$, and, since γ is strongly independent in $(\alpha_1,\dots,\alpha_p,\gamma)$, we know that $B \simeq \mathrm{susp}(\delta_{(\alpha_1,\dots,\alpha_p)})$, by Stabilization Theorem 3.1(b).

Let us now see that the homology map $d_i : \widetilde{H}_i(A \cup \widetilde{A}) \longrightarrow \widetilde{H}_i(\delta_{\tilde{\lambda}})$, induced by the inclusion map, is trivial, under the condition that $\tilde{\beta}_1$ and $\tilde{\beta}_2$ are independent in $\tilde{\lambda}$. Let K be the simplicial subcomplex of $\delta_{\tilde{\lambda}}$ consisting of the simplices indexed by those compositions, where $\tilde{\beta}_1$ is either in the same block as $\tilde{\beta}_2$ or in the block with a smaller index than the block containing $\tilde{\beta}_2$.

Clearly, $K \cong \delta_{(\bar{\alpha}_1,\dots,\bar{\alpha}_p,\bar{\beta}_1,\dots,\bar{\beta}_q,\bar{\gamma}_1,\dots,\bar{\gamma}_r)}$, where the set of resonances of the number partition $\bar{\lambda} = (\bar{\alpha}_1,\dots,\bar{\alpha}_p,\bar{\beta}_1,\dots,\bar{\beta}_q,\bar{\gamma}_1,\dots,\bar{\gamma}_r)$ is obtained from the set of resonances of $\tilde{\lambda}$ by adding the resonance $\bar{\beta}_1 = \bar{\beta}_2$ and everything which it implies together with the already existing resonances.

Since $\tilde{\beta}_1$ and $\tilde{\beta}_2$ are independent in $\tilde{\lambda}$, $\bar{\beta}_1 = \bar{\beta}_2$ is independent in $\bar{\lambda}$, hence, by Stabilization Theorem 3.1(a), K is contractible. On the other hand, $K \supseteq A \cup \widetilde{A}$, so the inclusion map can be factored $A \cup \widetilde{A} \xrightarrow{i^1} K \xrightarrow{i^2} \delta_{\tilde{\lambda}}$ and hence d_* factors as well $\widetilde{H}_*(A \cup \widetilde{A}) \xrightarrow{i^1} \widetilde{H}_*(K) \xrightarrow{i^2} \widetilde{H}_*(\delta_{\tilde{\lambda}})$. Since the middle term is 0, we conclude that $d_i = 0$. The last conclusion of the theorem now follows.   ∎

3.3. NONCANONICITY OF MAPS BETWEEN RESONANCES.   Recall that if $\mathrm{Res}\,\lambda = \mathrm{Res}\,\mu$, then $\delta_{\lambda} \simeq \delta_{\mu}$. Therefore, the simplicial complex $\delta_{\mathrm{Res}\,\lambda}$ is well-defined.

Whenever we have number partitions $\lambda > \mu$, there is an inclusion map $i(\mu, \lambda) : \delta_{\mu} \to \delta_{\lambda}$ which induces $i(\mu, \lambda)_* : \widetilde{H}_*(\delta_{\mu}) \to \widetilde{H}_*(\delta_{\lambda})$, and hence obviously $i(\mu, \lambda)_* : \widetilde{H}_*(\delta_{\mathrm{Res}\,\mu}) \to \widetilde{H}_*(\delta_{\mathrm{Res}\,\lambda})$. We conjecture, however, that $i(\mu, \lambda)_*$ depends on more than just the sets $\mathrm{Res}\,\lambda$ and $\mathrm{Res}\,\mu$, i.e., one cannot define a unique map $i(\mathrm{Res}\,\mu, \mathrm{Res}\,\lambda)_*$. More precisely:

CONJECTURE 3.7: *There exist number partitions* $\lambda, \mu, \tilde{\mu}$, *such that*

(1) $\lambda > \mu$, $\lambda > \tilde{\mu}$;

(2) $\text{Res}\,\mu = \text{Res}\,\tilde{\mu}$;

(3) *the homomorphisms of homology groups,* $i(\mu, \lambda)_*$ *and* $i(\tilde{\mu}, \lambda)_*$, *induced by the respective inclusion maps are nonisomorphic.*

For every $n \geq 1$, we define a partial order on the set $\{\text{Res}\,\lambda \mid l(\lambda) = n\}$ by saying that $\text{Res}\,\tilde{\lambda} \geq \text{Res}\,\lambda$ if and only if for some compositions α, resp. $\tilde{\alpha}$, of type λ, resp. $\tilde{\lambda}$, the set of resonances of $\tilde{\alpha}$ is a subset of the set of resonances of α. In such a case, a choice of the compositions α and $\tilde{\alpha}$ induces a map $\gamma(\tilde{\alpha}, \alpha): \delta_{\tilde{\lambda}} \to \delta_\lambda$, and further $\gamma(\tilde{\alpha}, \alpha)_*: \widetilde{H}_*(\delta_{\tilde{\lambda}}) \to \widetilde{H}_*(\delta_\lambda)$.

CONJECTURE 3.8: *The isomorphism type of the map* $\gamma(\tilde{\alpha}, \alpha)_*$ *depends not only on the actual number partitions* $\tilde{\lambda}$ *and* λ, *rather than their sets of resonances, but even on the choice of the pair of compositions* α *and* $\tilde{\alpha}$.

Note: It is easy to come up with examples of number partitions $\tilde{\lambda}$ and λ for which there are such pairs of compositions, which are nonisomorphic under the group of symmetries of the blocks of λ. For example, $\tilde{\lambda} = (a, b, c, d, d)$, $\lambda = (x, x, x, y, y)$, $\tilde{\alpha}_1 = \tilde{\alpha}_2 = [a, b, c, d, d]$, $\alpha_1 = [x, x, x, y, y]$, and $\alpha_2 = [x, y, y, x, x]$; the pairs $(\tilde{\alpha}_1, \alpha_1)$ and $(\tilde{\alpha}_2, \alpha_2)$ are nonisomorphic.

We would like to emphasize that in order to obtain an algorithm for computing the homology groups of the simplicial complexes $\delta_{\text{Res}\,\lambda}$, it is almost certainly essential to understand the maps $i(\mu, \lambda)_*$ and $\gamma(\tilde{\alpha}, \alpha)_*$, which, as Conjectures 3.7 and 3.8 seem to suggest, may be a rather nontrivial task.

4. Remaining questions and future perspectives

We think that understanding the maps $i(\mu, \lambda)_*$ and $\gamma(\phi(\tilde{\lambda}, \lambda))_*$ described in Subsection 3.3, combined with the type of arguments used in the proof of Theorem 3.6, would lead to further progress in the computation of the homology groups of the simplicial complexes δ_λ.

The following observation (the proof is left to the reader) is of use when one wants to compare two long exact sequences of a pair, as was done in Theorem 3.6.

PROPOSITION 4.1: *Let* $(\pi_1, \ldots, \pi_{l(\lambda)}) = \lambda > \mu$ *and* $(\tilde{\pi}_1, \ldots, \tilde{\pi}_{l(\tilde{\lambda})}) = \tilde{\lambda} > \tilde{\mu}$ *be number partitions, such that* $l(\lambda) = l(\tilde{\lambda})$ *and* $l(\mu) = l(\tilde{\mu})$. *We have a simplicial isomorphism of pairs* $(\delta_\lambda, \delta_\mu)$ *and* $(\delta_{\tilde{\lambda}}, \delta_{\tilde{\mu}})$ *induced by* $\pi_i \to \tilde{\pi}_i$, *for* $1 \leq i \leq l(\lambda)$ *(in which case, of course,* $\widetilde{H}_*(\delta_\lambda, \delta_\mu) = \widetilde{H}_*(\delta_{\tilde{\lambda}}, \delta_{\tilde{\mu}})$), *if and only if the following conditions are satisfied:*

(1) $\delta_{\tilde{\mu}}$ is the image of δ_{μ} under the map induced by $\pi_i \to \tilde{\pi}_i$;

(2) if there is a resonance $\sum_{i \in I} \pi_i = \sum_{j \in J} \pi_j$ in λ, then either we have

$$\mu \geq (\sum_{i \in I} \pi_i, \sum_{j \in J} \pi_j, \pi_{f_1}, \ldots, \pi_{f_t}),$$

where $\{f_1, \ldots, f_t\} = \{1, \ldots, l(\lambda)\} \setminus (I \cup J)$, or there is a resonance $\sum_{i \in I} \tilde{\pi}_i = \sum_{j \in J} \tilde{\pi}_j$ in $\tilde{\lambda}$;

(3) the same as (2) above, with λ and $\tilde{\lambda}$, as well as μ and $\tilde{\mu}$, interchanged.

EXAMPLE OF A COMPUTATION.   Let $\lambda = (3, 2, 2, 1)$, $\mu = (3, 3, 2)$, $\tilde{\lambda} = (5, 3, 3, 1)$, and $\tilde{\mu} = (5, 4, 3)$; the conditions of Proposition 4.1 are satisfied. Clearly, $\delta_{(3,3,2)}$ is contractible. By Stabilization Theorem 3.1(a) and Proposition 3.4 we have $\tilde{H}_*(\delta_{(5,3,3,1)}) = \mathbb{Z}_{(1)}$, and, by direct observation, $\delta_{(5,4,3)}$ is homeomorphic to S^1. Therefore we conclude that

$$\tilde{H}_*(\delta_{(3,2,2,1)}) = \tilde{H}_*(\delta_{(3,2,2,1)}, \delta_{(3,3,2)}) = \tilde{H}_*(\delta_{(5,3,3,1)}, \delta_{(5,4,3)}) = \mathbb{Z}_{(2)} \oplus \mathbb{Z}_{(1)},$$

where the last equality follows from the fact that the circle $\delta_{(5,4,3)}$ does not pass through the vertex indexed by the composition $(6, 6)$, hence the homomorphism of the homology groups $i_* : \tilde{H}_1(\delta_{(5,4,3)}) \to \tilde{H}_1(\delta_{(5,3,3,1)})$, induced by inclusion, is a zero map.

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